Kennedy Baptist College

Semester One Examination 2018
Question/Answer Booklet

MATHEMATICS
METHODS UNIT 3

Section Two:
Calculator-assumed

Student Name: Solutions

Teacher's Name: ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for paper: one hundred minutes

Material required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens(blue/black preferred), pencils(including coloured), sharpener, correction tape/fluid, erasers, ruler, highlighters

Special Items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations.

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

	Number of questions available	Number of questions to be attempted	Working time (minutes)	Marks available	Percentage of exam
Section One Calculator—free	9	9	50	52	35
Section Two Calculator—assumed	**13**	**13**	**100**	**96**	**65**
					100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2018.* Sitting this examination implies that you agree to abide by these rules.

2. Answer the questions according to the following instructions.

   Section Two: Write answers in this Question/Answer Booklet. Answer **all** questions.

   Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

   It is recommended that you **do not use pencil**, except in diagrams.

3. You must be careful to confine your responses to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. The Formula Sheet is **not** handed in with your Question/Answer Booklet.

Section Two: Calculator–Assumed 96 marks

This section has **thirteen (13)** questions. Attempt **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Working time: 100 minutes

Question 10 **(5 marks)**

A sector of a circle has area *K*, given by $K = \frac{1}{2}r^2\sin\theta$, where *r* is the radius of the circle, and θ is the central angle.

(a) Use the incremental formula to approximate the increase in *K*, as θ changes from $\frac{\pi}{4}$ to $0{\cdot}3\pi$ in a circle of radius 4 cm. (3 marks)

$\delta\theta = 0.3\pi - \frac{\pi}{4}$

$\frac{dK}{d\theta} = 8\cos\theta$ at $r=4$, $\delta\theta = \frac{\pi}{20}$ or 0.05π ✓

$\delta K = \frac{dK}{d\theta} \times \delta\theta \rightarrow \delta K = 8\cos\theta\left(\frac{\pi}{20}\right)$ ✓

$\delta K = 8\cos\left(\frac{\pi}{4}\right)\left(\frac{\pi}{20}\right)$

$= \frac{\sqrt{2}\pi}{5}$ or 0.889 ✓

(b) Determine the exact increase in K and hence determine the percentage error in your approximation from (a). Give your answer to one decimal place. (2 marks)

$K(0.3\pi) = 6.4721$

$K\left(\frac{\pi}{4}\right) = 5.6569$

Error $= 0.8153$ ✓

% Error $= \frac{0.889 - 0.8153}{0.8153} \times 100$

$= 9.0\%$ ✓ (1d.p)

Question 11 (7 marks)

A curve has equation $y = ax^3 - bx^2 + cx - 9$.
There is a stationary point at $(-3, 0)$.
There is a point of inflection at $x = -\frac{5}{3}$.

Determine a, b and c. Show your working. (7 marks)

at $(-3,0) \rightarrow \quad -27a - 9b - 3c - 9 = 0$ ✓

$\frac{dy}{dx} = 3ax^2 - 2bx + c$ ✓

stationary point at $x = -3 \rightarrow \frac{dy}{dx} = 0$

$27a + 6b + c = 0$ ✓

$\frac{d^2y}{dx^2} = 6ax - 2b$ ✓

p.o.i at $x = -\frac{5}{3} \rightarrow \frac{d^2y}{dx^2} = 0$

$-10a - 2b = 0$ ✓

$$\left.\begin{array}{ll} ① & -27a - 9b - 3c - 9 = 0 \\ ② & 27a + 6b + c = 0 \\ ③ & -10a - 2b = 0 \end{array}\right\} \text{ solve with classpad}$$

$\therefore a = 1, \; b = -5, \; c = 3$ ✓✓

Question 12 **(6 marks)**

A function f has $f'(x) = 2\sin\frac{x}{2}$.

(a) Determine the equation of f given that it passes through the point $\left(\frac{\pi}{2}, 4 - 2\sqrt{2}\right)$. (3 marks)

$$f(x) = \int 2\sin\frac{x}{2}\,dx = -4\cos\frac{x}{2} + c \checkmark$$

$$f\left(\frac{\pi}{2}\right) = -4\cos\frac{\pi}{4} + c = 4 - 2\sqrt{2} \checkmark$$

$$-4\left(\frac{1}{\sqrt{2}}\right) + c = 4 - 2\sqrt{2}$$

$$c = 4$$

$$\therefore f(x) = -4\cos\frac{x}{2} + 4 \checkmark$$

(b) State the next value of x where the graph will have the same height (ie. $4 - 2\sqrt{2}$). Explain your reasoning. (3 marks)

$$-4\cos\frac{x}{2} + 4 = 4 - 2\sqrt{2} \checkmark$$

$$\cos\frac{x}{2} = \frac{\sqrt{2}}{2} \checkmark$$

$$\frac{x}{2} = \frac{\pi}{4}, \frac{7\pi}{4}$$

$$\therefore x = \frac{\pi}{2}, \frac{7\pi}{2} \checkmark$$

next time is $x = \frac{7\pi}{2}$

Question 13 (4 marks)

The exponential series can be expressed as:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.$$

(a) Use the first five terms of the series to evaluate e as a fraction. (Hint: $e = e^1$) (2 marks)

$$e^1 = 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} \checkmark$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}$$

$$= \frac{65}{24} \text{ or } 2\frac{17}{24} \checkmark$$

(b) Use the series to show that the derivative of e^x is e^x. (2 marks)

$$\frac{d}{dx} e^x$$

$$= \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

$$= 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots \checkmark$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$= e^x \checkmark$$

Question 14 (6 marks)

A function $M(x)$ has $M'(x) = 3x^2 - kx$,where k is a constant.
A stationary point occurs at (6, 1).

(a) Calculate the value of k. stationary point when $M'(x) = 0$ (2 marks)

$3x^2 - kx = 0$

$108 - 6k = 0$ at $x = 6$ ✓

$\therefore k = 18$ ✓

(b) Determine the value of the y-intercept on the graph of M. (4 marks)

$M'(x) = 3x^2 - 18x$

$M(x) = x^3 - 9x^2 + c$ ✓

at $(6,1) \rightarrow 1 = 216 - 324 + c$ ✓

$c = 109$ ✓

$\therefore$ y-intercept $= (0, 109)$ ✓

* accept 109

Question 15 (15 marks)

75% of adults in a certain town graduated from high school.

(a) Thirty adults are randomly selected using the council records of the town.

(i) Calculate the probability that twenty five of these adults graduated from high school. State the probability distribution, and any parameters associated with that distribution. (3 marks)

$X \sim Bin(30, 0.75)$ ✓

$P(X=25) = 0.1047$ ✓ ✓

(ii) Given that at least twenty five graduated, find the probability that more than twenty eight graduated. (3 marks)

$$\frac{P(X \geq 29)}{P(X \geq 25)} = \frac{0.00196}{0.20260}$$ ✓ ✓

$= 0.0097$ ✓

(b) How many adults need to be randomly selected so that the probability that at least ten graduated is at least 99%? (3 marks)

$Y \sim Bin(n, 0.75)$ ✓

$P(Y \geq 10) \geq 0.99$ ✓

$n = 19$ ✓

(trial and error on classpad)

The Smith family consists of eleven people. Five of the Smiths graduated from high school.

(c) (i) If three of the Smiths attend a concert together, find the probability that at least one of them graduated from high school. (3 marks)

$$P(G \geq 1) = \frac{{}^5C_3\,{}^6C_0 + {}^5C_2\,{}^6C_1 + {}^5C_1\,{}^6C_2}{{}^{11}C_3}$$ ✓ ✓

$$= \frac{10 + 60 + 75}{165}$$

$$= 0.87879$$ ✓

OR $1 - P(G=0)$

$$= 1 - \frac{{}^6C_3}{{}^{11}C_3}$$

$$= 0.8788$$

(ii) If Peter and Sally are part of this family, find the probability that one, but not both of them, graduated from high school. (3 marks)

Peter or Sally

$$\frac{{}^5C_1\,{}^6C_1}{{}^{11}C_2} = \frac{30}{55}$$ ✓ ✓

$$= \frac{6}{11} \text{ OR } 0.5455$$ ✓

Question 16 (6 marks)

(a) Solve $\sin 2x + \cos x = 0$ for $0 \le x \le 2\pi$. (2 marks)

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ ✓ ✓ ✓ ✓

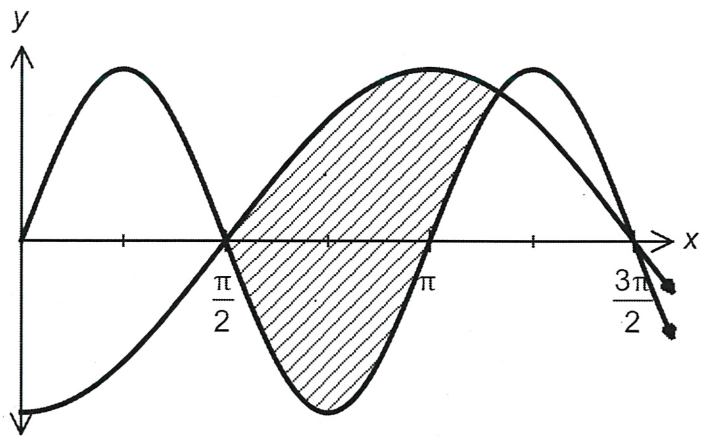
* Solve on classpad

Hence, or otherwise,

(b) determine the points of intersection of $y = \sin 2x$ and $y = -\cos x$ for $0 \le x \le 2\pi$. (2 marks)

$\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{7\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{11\pi}{6}, \frac{-\sqrt{3}}{2}\right)$ ✓ ✓ ✓ ✓

(c) The graphs of $y = \sin(2x)$ and $y = -\cos(x)$ are drawn below. Determine the area of shaded region. (2 marks)



$A = \int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} (-\cos x - \sin 2x)\, dx$ ✓

$= 2.25 \text{ units}^2$ ✓

OR $\frac{9}{4} \text{ units}^2$

Question 17 (5 marks)

Consider the function $y = \dfrac{x+1}{x-1}$

(a) Determine the x–intercept(s) of the function. (1 mark)

$x = -1$ or $(-1, 0)$ ✓

(b) (i) Determine $\dfrac{dy}{dx}$. (1 mark)

$$\frac{dy}{dx} = -\frac{2}{(x-1)^2} \checkmark$$

(ii) Prove the conjecture; "This function has no turning points." (1 mark)

$$\frac{dy}{dx} = \frac{-2}{(x-1)^2} \neq 0 \text{ for } x \in R$$

$\therefore$ True ✓

(c) (i) Determine $\dfrac{d^2y}{dx^2}$. (1 mark)

$$\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3} \checkmark$$

(ii) Prove the conjecture; "This function has no points of inflection." (1 mark)

$$\frac{d^2y}{dx^2} = \frac{4}{(x-1)^3} \neq 0 \text{ for } x \in R$$

$\therefore$ True ✓

Question 18 (13 marks)

A particle is undergoing rectilinear motion. The velocity of the particle is given by $v = 2t^2 - 5t + 3$ where t is time in seconds. Displacement is in metres.

(a) Determine the particle's initial velocity. (1 mark)

$v(0) = 3 m/s$ ✓

(b) Calculate when the particle is stationary. (3 marks)

Stationary when $V = 0$

$2t^2 - 5t + 3 = 0$ ✓

$t = 1$ ✓ and $t = 1.5$ ✓

(c) (i) Determine an expression for the acceleration. (1 mark)

$a = \frac{dv}{dt}$

$a = 4t - 5$ ✓

(ii) Hence, or otherwise, determine when the velocity is a minimum. (2 marks)

V is minimum when $a = 0$

$4t - 5 = 0$ ✓

$t = 1.25$ seconds ✓

or $\frac{5}{4}$ seconds

The particle has a displacement, x metres, from the origin O on the line.
It is initially at the origin.

(d) Determine the displacement when $t = 3$. (3 marks)

$x = \int v \, dt$

$x = \frac{2}{3}t^3 - \frac{5}{2}t^2 + 3t + c$

at $t=0 \rightarrow x(0) = 0 \rightarrow c = 0$ ✓

$x = \frac{2}{3}t^3 - \frac{5}{2}t^2 + 3t$ ✓

$x(3) = 4.5$ m ✓

(e) Determine the distance travelled in the first 3 seconds to the nearest cm. (2 marks)

Distance $= \int_0^3 |v(t)| \, dt$

$= \int_0^3 |2t^2 - 5t + 3| \, dt$ ✓

$= 4.58$ m ✓

(f) Explain why the answers to (d) and (e) are different. (1 mark)

Distance travelled $\neq$ Displacement ✓

OR because the particle stops and changes direction at $t=1$ and $t=1.5$.

Question 19 (6 marks)

A forest fire is estimated to be spreading at a rate of 5% per hour. The area A, in ha, covered by the fire at any time t, in hours since the fire was discovered, is defined by $\frac{dA}{dt} = 0{\cdot}05A$.
At the moment of discovery, the area covered was 0.6 ha.

(a) What will be the area of the fire ten hours after it was discovered?
Give your answer correct to two decimal places. (3 marks)

$A = A_0 e^{kt}$, $k = 0.05$ ✓

$A = 0.6e^{0.05t}$ ✓

at $t = 10 \rightarrow A = 0.6e^{0.05(10)}$

$= 0.99$ Ha ✓

(b) When would the fire cover an area of 5 ha?
Give your answer to the nearest hour. (3 marks)

$5 = 0.6e^{0.05t}$ ✓

$\therefore t = 42.4$ hours ✓

$t = 42$ hours (nearest hour) ✓

Question 20 (10 marks)

Consider f where $f(x) = \cos x + \sin x$ where $0 \leq x \leq 2\pi$.

(a) Determine:

(i) $f'(x)$. (1 mark)

$f'(x) = -\sin x + \cos x$ ✓

(ii) $f''(x)$. (1 mark)

$f''(x) = -\cos x - \sin x$ ✓

(b) State the exact maximum value of f over the given domain.
Prove it is the maximum using the second derivative test. (4 marks)

Maximum when $f'(x) = 0$

$-\sin x + \cos x = 0$

$x = \frac{\pi}{4}$ ✓ $\rightarrow$ $f(x) = \sqrt{2}$ ✓

$f''(\frac{\pi}{4}) < 0$ ✓ $\therefore$ maximum ✓

(c) Identify any points of inflection over the domain. Give the exact answer. (4 marks)

P.o.i when $f''(x) = 0$

$-\cos x - \sin x = 0$ ✓

$x = \frac{3\pi}{4}$ ✓ or $\frac{7\pi}{4}$ ✓

$\therefore \left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$ ✓

Question 21 (4 marks)

A long gas pipe is closed, but gas continues to flow out. The rate of flow $\frac{dF}{dt}$, in litres per second, is given by $\frac{dF}{dt} = 10 - \frac{t}{20}$.

(a) What is the initial rate of flow? (1 mark)

at $t = 0 \rightarrow \frac{dF}{dt} = 10$ L/sec ✓

(b) How many seconds does it take till the flow stops? (1 mark)

$10 - \frac{t}{20} = 0$

$t = 200$ seconds ✓

OR 3 mins 20 secs

(c) How much gas flows in total after the pipe is closed? (2 marks)

$\int_0^{200} \left(10 - \frac{t}{20}\right) dt$ ✓

$= 1000$ L ✓

Question 22 (9 marks)

A drop of oil is spreading on a glass surface. The region covered is circular in shape, and the radius, r cm, of the circle is given as a function of time, t seconds.

$$r = -e^{-t} + 4$$

(a) Find the rate at which the radius is increasing when:

(i) $t = 4$. (2 marks)

$r' = e^{-t}$ ✓

$r'(4) = e^{-4} = 0.018$ cm/s ✓

(ii) $t = 5$. (1 mark)

$r'(5) = e^{-5} = 0.0067$ cm/s ✓

(b) Use your knowledge of exponential functions and its graph, or derivatives, to determine when the radius is increasing at its fastest rate. (2 marks)

the graph of $r'(t)$ is decreasing ✓

OR the graph of $r''(t)$ has the greatest slope at $t=0$.

$\therefore$ the radius is increasing at its highest rate at $\underline{\underline{t=0}}$. ✓

(c) At what rate is the area of the circle increasing when $t = 4$? (4 marks)

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ ✓ $\left(\frac{dr}{dt} = e^{-t} \text{ from (a)}\right)$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= 2\pi r \times e^{-t}$ ✓

other Method:

$A = \pi r^2$

$A = \pi(-e^{-t}+4)^2$ ✓

$\frac{dA}{dt} = 2\pi(-e^{-t}+4)(e^{-t})$ ✓

at $t=4$ ✓sub

$\frac{dA}{dt} = 0.458$ cm²/s ✓

at $t=4 \rightarrow \frac{dA}{dt} = 2\pi(-e^{-4}+4)(e^{-4}) = 0.458$ cm²/s

✓ ✓

END OF QUESTIONS